One Dimensional Conductive Heat Transfer of a Uniform Rod by using Finite Volume Method

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Abstract— *In this paper, finite volume method has been used to investigate one dimensional conductive heat transfer throw a uniform insulated uniform rod. Then the step by step procedures of this numerical solution is described where tri-diagonal matrix algorithm is applied to solve the system of our algebraic equations. Finally, to check the accuracy of our method a comparison between the numerical solution obtained by finite volume techniques and exact solution is presented.*

Keywords—*Computational Fluid Dynamics, 1D Heat Conduction, Finite Volume Method, Discretized Equations, Tri-Diagonal Matrix Algorithm*

I. Introduction

Nowadays, to investigate the mass, momentum, and heat transfer phenomena, computational fluid dynamics (CFD) study numerical solutions which is associated with the phenomena of thermodynamics, turbo machinery, chemical manufacturing, power generation, weather simulation, biological engineering, meteorology, aerospace, reaction chemistry, predicting fluid flow and heat transfer. Though the fundamental source of almost all CFD problems is Navier–Stokes equations which describe many single-phase (liquid or gas, but not both) fluid flows.

Recently, due to the revolution of computer technology, abundant computational grid techniques have been developed which is very efficient to solve numerous engineering problems [1-2]. Among these numerical grid techniques, finite difference method (FDM) is used as a common numerical technique to solve numerous engineering problems [3]. Again, finite element method (FEM) is another kind of commonly used numerical technique which has been applied to solve many heat transfer problems [4]. A generalized transfer equation for a dependent variable ϕ which can be mass, concentration, heat and momentum is given by Patankar [5] where discretization technique is applied for CFD analysis. Moreover, the finite volume method (FVM) is becoming most popular numerical technique which is commonly used in commercial CFD software such as COMSOL Multiphysics [6]. Though to solve any problem these three methods has its own merits besides its demerits, FVM technique is one of the utmost flexible and multipurpose technique to solve CFD. In this paper, we have described an engineering problem by the point of view of FVM. Due to its wide-spread popularity in CFD, this method has been applied previously in different mechanical engineering problem as like thermoelastic and linear elastic problems [7-9].

The rest of our paper is prepared as follows. A short review of FVM with the help of Tri-Diagonal Matrix Algorithm TDMA is given in Section 2. In Section 3, the numerical solutions obtained by this technique is given where a comparison between exact and our numerical solutions is also described. Finally, in Section 5 we have concluded this the paper.

II. FINITE VOLUME METHOD

Consider the steady state diffusion of a property T in a one-dimensional domain defined in Fig. 1. The process is governed by

$$\frac{d}{dx}\left(K\frac{dT}{dx}\right)+S=0 \tag{1}$$

where K is the thermal connectivity, T is the temperature and S is the source term (i.e., the rate of heat generation per unit volume).

A. Grid Generation

In the finite volume method first of all, we have to divide the domain into discrete control volumes. Let us consider a number of nodal points in the domain space from *A* to *B*. The boundaries (or faces) of control volumes are positioned mid-way between adjacent nodes. Thus each node is surrounded by a control volume or cell. It is common practice to set up control volumes near the edge of the domain in such a way that the physical boundaries coincide with the control volume boundaries. At this point it is appropriate to establish a system of notation that can be used in future developments.

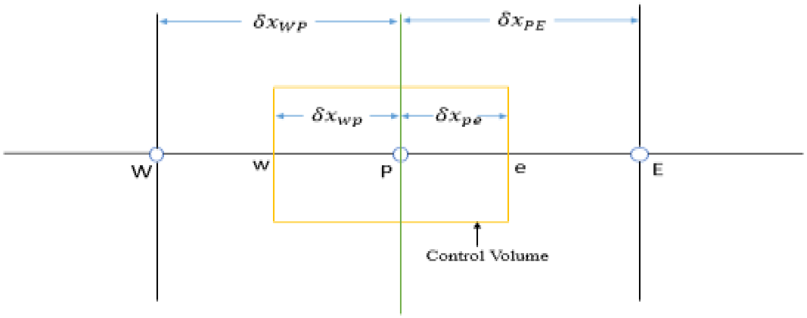

Fig. 1: Grid point generation for a one dimensional heat conduction problem

A general nodal point is identified by P and its neighbour's nodes to the west and east, are represented by W and E respectively. The east side control volume face is mentioned by e and the west side face of the control volume is mentioned to by w. The distances between the nodes *P* and *E, and* between nodes *W* and *P* are represented by δx_{PE} and δx_{WP} respectively. In the same way, distances between point *P* and face *e* is denoted by δx_{Pe} *and* distances between face *w* and point *P* is denoted by δx_{wP}.

B. Discretization

To discretize the governing equation, we will integrate the governing equation (1) over the control volume at its nodal point *P*.

$$\int_{\Delta V} \frac{d}{dx}\left(K \frac{dT}{dx}\right) dV + \int_{\Delta V} S \, dV = 0$$

Then we get,

$$\left(KA \frac{dT}{dx}\right)_e - \left(KA \frac{dT}{dx}\right)_w + \bar{S}\,\Delta V = 0 \tag{2}$$

where *A* is the cross-sectional area of the control volume face, ΔV is the volume and $\bar{S}$ is the average value of source *S* over the control volume. This is a very attractive feature of this finite volume method that the discretized equation has a clear physical interpretation. Equation (02) states that the diffusive flux of T leaving the east face minus the diffusive flux of T entering the west face is equal to the generation of T over the control volume. In order to derive useful forms of the discretized equations, the interface diffusion coefficient K and the gradient $\frac{dT}{dx}$ at west w and east e are required.

In a uniform grid linearly interpolated values for K_e and K_w are given by

$$K_e = \frac{K_P + K_E}{2} \tag{3a}$$

$$K_w = \frac{K_P + K_W}{2} \tag{3b}$$

And the diffusive flux terms are evaluated as

$$\left(KA \frac{dT}{dx}\right)_e = K_e A_e \left(\frac{T_E - T_P}{\delta x_{PE}}\right) \tag{4}$$

$$\left(KA\,\frac{dT}{dx}\right)_w = K_w A_w \left(\frac{T_P - T_W}{\delta x_{WP}}\right) \quad (5)$$

For a practical situations, the source term S may be a function of the dependent variable. In such cases the finite volume method approximates the source term by means of a linear form:

$$\bar{S}\,\Delta V = S_u + S_P T_P \quad (6)$$

By substituting the values of equations (4), (5) and (6) into equation (2), we get

$$K_e A_e \left(\frac{T_E - T_P}{\delta x_{PE}}\right) - K_w A_w \left(\frac{T_P - T_W}{\delta x_{WP}}\right) + (S_u + S_P T_P) = 0 \quad (7)$$

By rearranging equation (7), we can write

$$\left(\frac{K_e\,A_e}{\delta x_{PE}} + \frac{K_W\,A_W}{\delta x_{WP}} - S_P\right) T_P = \left(\frac{K_W\,A_W}{\delta x_{WP}}\right) T_W + \left(\frac{K_e\,A_e}{\delta x_{PE}}\right) T_E + S_u \quad (8)$$

By representing the coefficient of T_P, T_W and T_E as a_P, a_W and a_E then the equation (8) can be written as

$$a_P T_P = a_W\,T_W + a_E\,T_E + S_u \quad (9)$$

where

$a_W = \dfrac{K_w\,A_w}{\delta x_{WP}}$	$a_E = \dfrac{K_e\,A_e}{\delta x_{PE}}$	$a_P = a_E + a_W - S_P$

Again, the values of S_P and S_u can be obtained from the source nodal equation by

$$\bar{S}\,\Delta V = S_u + S_P T_P \quad (10)$$

When we will get the discretized equations from equation (9), to solve a problem we have to set up at each of the nodal points. The resulting system of linear algebraic equations will be solved to obtain the distribution of the property T at each nodal points. There are different types of matrix solution technique exist which can be used to solve the system of linear algebraic equations. From this we will use Tri-Diagonal Matrix Algorithm (TDMA) [10].

III. EXPERIMENT AND DISCUSSION

Consider an insulated rod whose length is $1m$ (Fig. 2) and this is a source free heat conduction system. The two ends of this insulated rod maintained at constant temperature as like $200°C$ and $600°C$ respectively. The thermal conductivity K of this rod is equals to $1000\ W/m.K$ and the cross-sectional area A is $20 \times 10^{-3}\ m^2$. The main goal of this work is to calculate the steady state temperature distribution by using our control volume method.

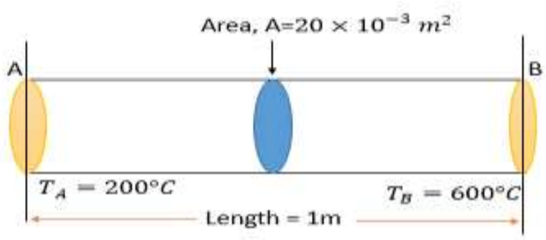

Fig. 2: A uniform insulated rod of $1m$ length

A. *Numerical Solution by FVM*

First of all, we have to divide the whole rod into some control volume. In this work, we will calculate the temperature distribution in five internal point. That is, the difference between two control volumes is $0.2\ m$ $\left(\text{since total length of the rod is 1m, so } \delta x = \frac{1}{5} m = 0.2\ m\right)$.

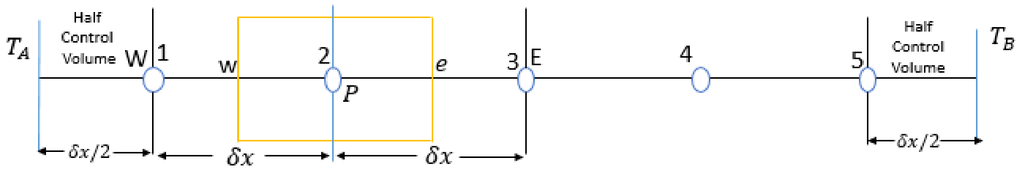

Fig. 3: Grid point generation for total domain with control volume and half control volume

For our given problem (one-dimensional heat conduction) governing equation is

$$\frac{d}{dx}\left(K\,\frac{dT}{dx}\right) = 0 \tag{11}$$

Since source term $S = 0$, so S is absent.
To get the discretization equation, we have to integrate equation (11) over the control volume. After implying (which has already described in section II) we get as like equation (8),

$$\left(\frac{K_e\,A_e}{\delta x_{PE}} + \frac{K_w\,A_w}{\delta x_{WP}}\right)T_P = \left(\frac{K_w\,A_w}{\delta x_{WP}}\right)T_W + \left(\frac{K_e\,A_e}{\delta x_{PE}}\right)T_E \tag{12}$$

Here, source term $S = 0$, so S_P and S_u are absent. So, equation (12) can be written as,

$$a_P T_P = a_W\ T_W + a_E\ T_E \tag{13}$$

where

$$a_W = \frac{K_w\,A_w}{\delta x_{WP}},\quad a_E = \frac{K_e\,A_e}{\delta x_{PE}}\ ,\quad a_P = a_E + a_W \tag{14}$$

In this problem, the given data are

$L = 1m$ $\delta x_{WP} = \delta x_{PE} = \delta x = \frac{1}{5}m = 0.2\ m$	$K_e = K_w = K$ $= 1000\ W/m.K$	$A_w = A_e = A = 20 \times 10^{-3}\ m^2$	$T_A = 200°C$ $T_B = 600°C$

By using these values into equation (14), we get $a_W = a_E = 100$ and $a_P = 200$. Again by using the value of a_W, a_E and a_P into equation (13), we get the discretized equation for the nodal point 2, 3, and 4 as,

$$\left.\begin{aligned} 200\ T_2 &= 100\ T_1 + 100\ T_3 \\ 200\ T_3 &= 100\ T_2 + 100\ T_4 \\ 200\ T_4 &= 100\ T_3 + 100\ T_5 \end{aligned}\right\} \tag{15}$$

On the other hand, at nodal point 1 which is a half control volume. For this reason equation (7) will be (where $S = 0$),

$$KA\left(\frac{T_E - T_P}{\delta x}\right) - KA\left(\frac{T_P - T_A}{\delta x/2}\right) = 0$$

Then,

$$\left(\frac{KA}{\delta x} + \frac{KA}{\delta x/2}\right)T_P = 0 \times T_W + \left(\frac{KA}{\delta x}\right)T_E + \left(\frac{KA}{\delta x/2}\right)T_A$$

which can be written as,

$$a_P T_P = a_W\ T_W + a_E\ T_E + S_u \tag{16}$$

where

$$a_W = 0,\quad a_E = \frac{KA}{\delta x},\quad S_P = \frac{KA}{\delta x/2},\quad a_P = a_E + S_P,\quad \text{and}\quad S_u = \frac{KA}{\delta x/2}T_A \tag{17}$$

By using the values of $a_W, a_E, a_P, \delta x, K, A$ and T_A, we get $a_E = 100, S_P = 200, S_u = 40000$ and $a_P = 300.$
Then the equation (16) can be written as,

$$300\ T_1 = 100\ T_2 + 40000 \tag{18}$$

which is the discretized equation for the nodal point 1. Similarly, at nodal point 5 which is a half control volume, the equation

(7) will be (where $S = 0$),

$$KA\left(\frac{T_B - T_P}{\delta x/2}\right) - KA\left(\frac{T_P - T_W}{\delta x}\right) = 0$$

Then, $$\left(\frac{KA}{\delta x} + \frac{KA}{\delta x/2}\right)T_P = \left(\frac{KA}{\delta x}\right)T_W + 0 \times T_E + \left(\frac{KA}{\delta x/2}\right)T_B$$

which cab be written as,

$$a_P T_P = a_W\, T_W + a_E\, T_E + S_u \tag{19}$$

where

$$a_W = \frac{KA}{\delta x}, \quad a_E = 0, \quad S_P = \frac{KA}{\delta x/2}, \quad a_P = a_w + S_P, \quad \text{and} \quad S_u = \frac{KA}{\delta x/2}T_B \tag{20}$$

By using the values of $a_W, a_E, a_P, \delta x, K, A$ and T_B, we get $a_W = 100, S_P = 200, S_u = 120000$ and $a_P = 300$. Then the equation (19) can be written as,

$$300\, T_5 = 100\, T_4 + 120000 \tag{21}$$

which is the discretized equation for the nodal point 5.

Therefore, the set of discretized equations are:

$$\left.\begin{aligned} 300\, T_1 &= 100\, T_2 + 40000 \\ 200\, T_2 &= 100\, T_1 + 100\, T_3 \\ 200\, T_3 &= 100\, T_2 + 100\, T_4 \\ 200\, T_4 &= 100\, T_3 + 100\, T_5 \\ 300\, T_5 &= 100\, T_4 + 120000 \end{aligned}\right\} \tag{22}$$

This system of algebraic linear equation can be solved by different method such as (tri-diagonal matrix algorithm, Gaussian elimination method, etc.). But this system of equation can be solved easily by MATLAB. Then we get our temperature distribution as,

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 240 \\ 320 \\ 400 \\ 480 \\ 560 \end{bmatrix}$$

B. Exact Solution

Our governing equation is $\frac{d}{dx}\left(K\frac{dT}{dx}\right) = 0$ where boundary conditions, $T_A = 200°C$ at $x = 0$ and $T_B = 600°C$ $x = 1$. By simplifying this we get our general solution,

$$T = Cx + B \tag{23}$$

where *B* and *C* are integral constant. By applying the boundary condition we get $B = 200$ and $C = 400$, so the complete solution is,

$$T = 400x + 200 \tag{24}$$

For different values of x, we will get the temperature distribution from equation (24) at different nodal points which is same as our numerical solution (in Table-1).

Table-1
Comparison table between numerical and exact solutions

Nodal Point	Value of x (m)	Exact Solution	Control Volume Method (TDMA)	Error (%)
1	0.1	240	240	0
2	0.3	320	320	0
3	0.5	400	400	0

4	0.7	480	480	0
5	0.9	560	560	0

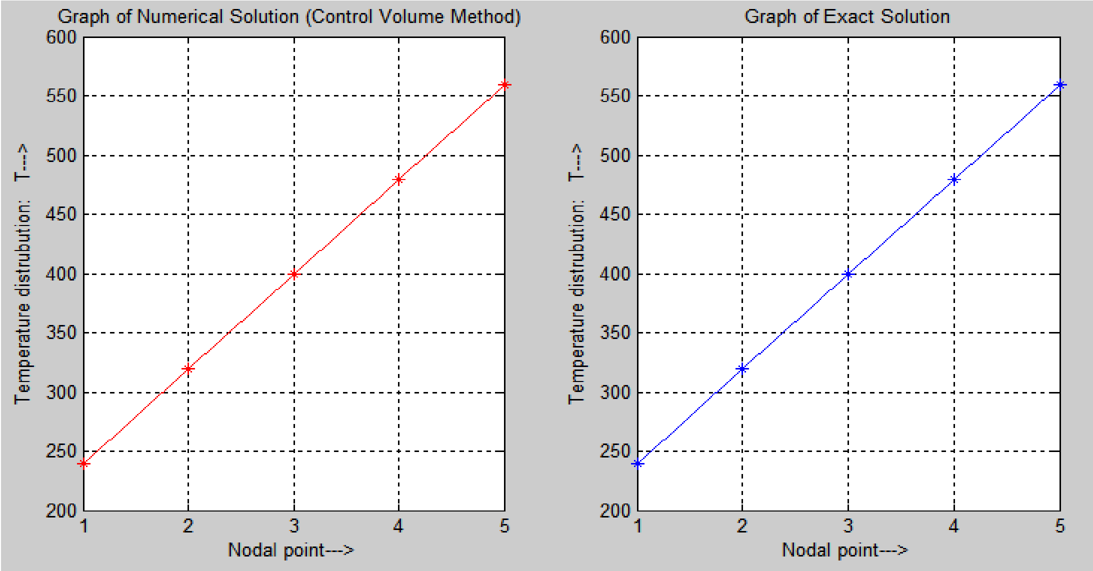

Fig. 4: Graphical comparison between control volume method and exact/analytic solution

IV. CONCLUSIONS

In this work, to investigate one dimensional conductive heat transfer in a uniform insulated rod, we have been used the finite volume method (FVM). Obtaining discretized equation by FVM, we have applied Tri-Diagonal Matrix Algorithm (TDMA) method. Finally, with the help of MATLAB R2014a, we have represented a tabular and a graphical comparison between our numerical solution and exact solution. The FVM give an outstanding results which has no error with respect to exact solution. That is, this method is very effective, accurate, reliable and easier to appliance in MATLAB compared to the other exorbitant methods.

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